DE-BROGLIE HYPOTHESIS
PHASE VELOCITY
GROUP VELOCITY

Light is . . .

- Initially thought to be waves
  - They do things waves do, like diffraction and interference
  - Wavelength – frequency relationship $c = \lambda f$
- Planck, Einstein, Compton showed us they behave like particles (photons)
  - Energy comes in chunks
  - Wave-particle duality: somehow, they behave like both $E = hf$
- Photons also carry momentum
  - Momentum comes in chunks

$$p = E/c = hf/c = h/\lambda$$

$$\boxed{p\lambda = h}$$

Electrons are . . .

- They act like particles
  - Energy, momentum, etc., come in chunks
- They also behave quantum mechanically
- Is it possible they have wave properties as well?

The de Broglie Hypothesis

- Two equations that relate the particle-like and wave-like properties of light

$$E = hf$$

$$\lambda p = h$$

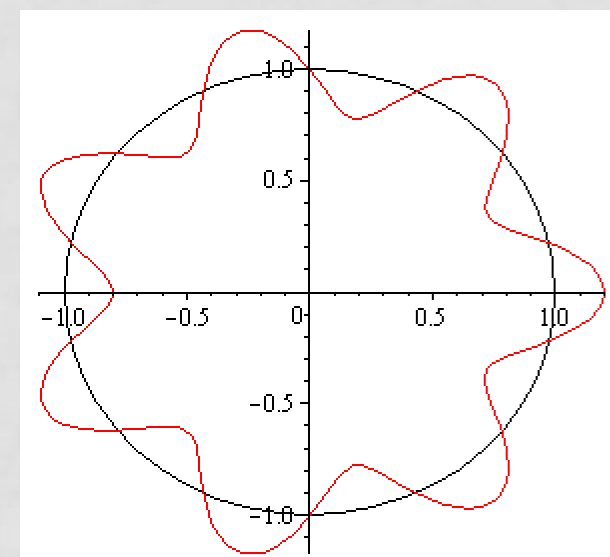
1924 – Louis de Broglie postulated that these relationships apply to electrons as well

- Implied that it applies to other particles as well
- de Broglie could simply explain the Bohr quantization condition
  - Compare the wavelength of an electron in hydrogen to the circumference of its path

$$L = n\hbar = m_e vr = pr = \frac{hr}{\lambda} = \frac{2\pi\hbar r}{\lambda}$$

cancel □

$$n\lambda = 2\pi r = C$$



Integer number of wavelengths fit around the orbit

Measuring wave properties of electrons

$$\lambda p = h$$

$$E = \tfrac{1}{2} m v^2 = \frac{p^2}{2m} = \frac{h^2 c^2}{2mc^2 \lambda^2} = \frac{\left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)^2 \left(3.00\times10^{8}\,\text{m/s}\right)^2}{2\left(0.511\times10^{6}\ \text{eV}\right)\lambda^2}$$

$$= \frac{1.504\times10^{-18}\ \text{eV}\cdot\text{m}^2}{\lambda^2} = 1.504\ \text{eV}\left(\frac{\text{nm}}{\lambda}\right)^2$$

For atomic separations, want distances around 0.3 nm → energies of 10 or so eV

How can we measure these wave properties?

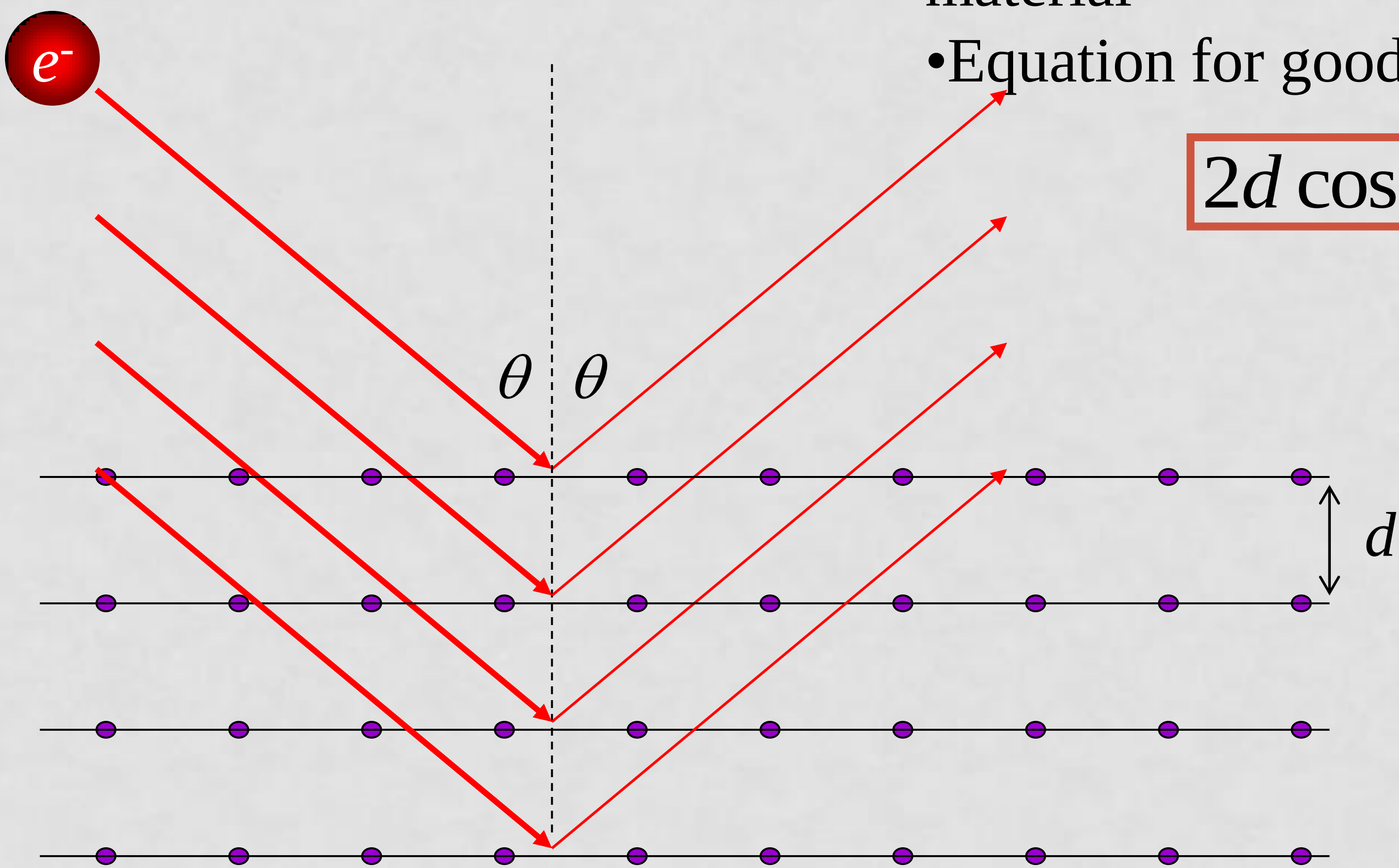
- Scatter off crystals, just like we did for X-rays!
- Complication: electrons change speed inside crystal
  - Work function ϕ increases kinetic energy in the crystal
  - Momentum increases in the crystal
  - Wavelength changes

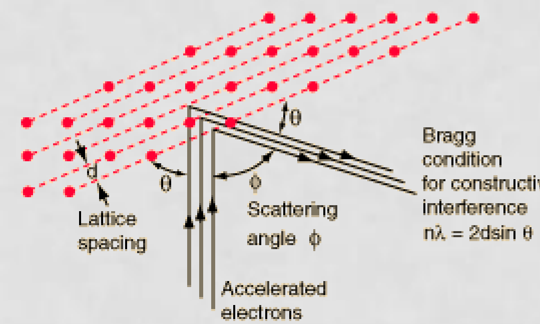
The Davisson-Germer Experiment

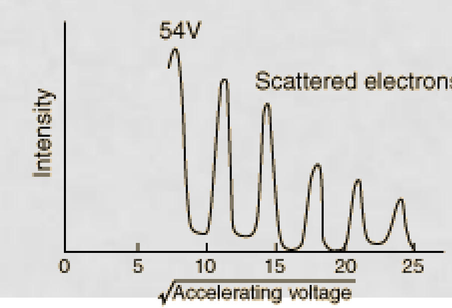
Same experiment as scattering X-rays, except

- Reflection probability from each layer greater
  - Interference effects are weaker
- Momentum/wavelength is shifted inside the material
- Equation for good scattering identical

$$2d\cos\theta = m\lambda$$



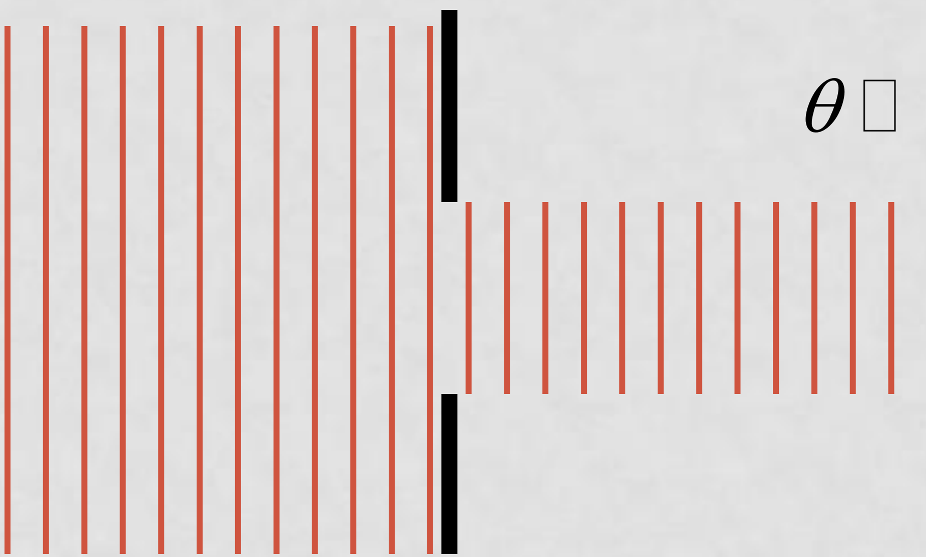




•Whenever waves encounter a barrier, they get diffracted, their direction changes

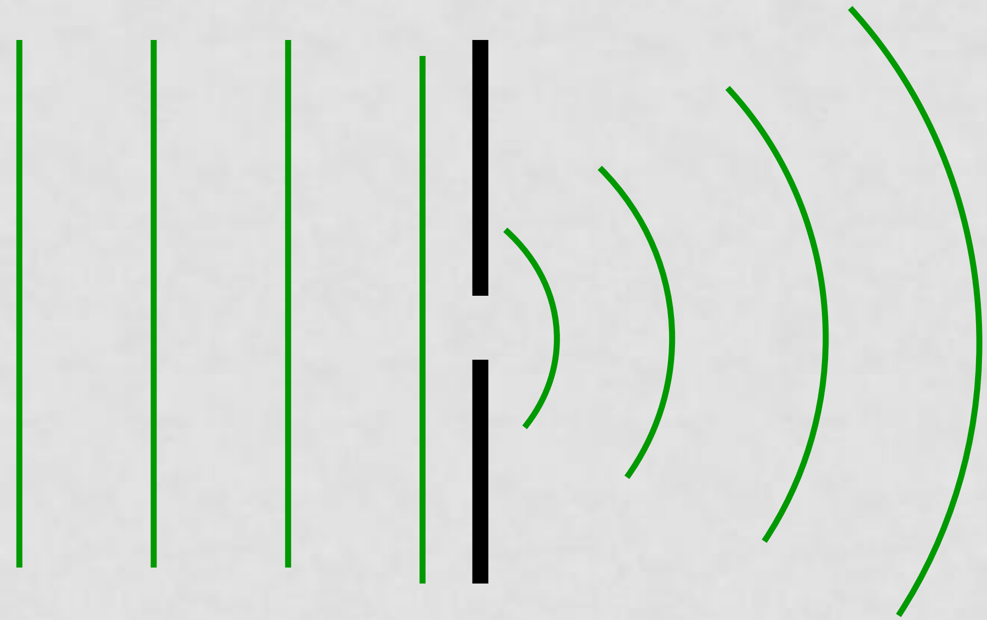
•If the barrier is much *larger* then the waves, the waves change direction very little

•If the barrier is much *smaller* then the waves, then the effect is enormous, and the wave diffracts a lot

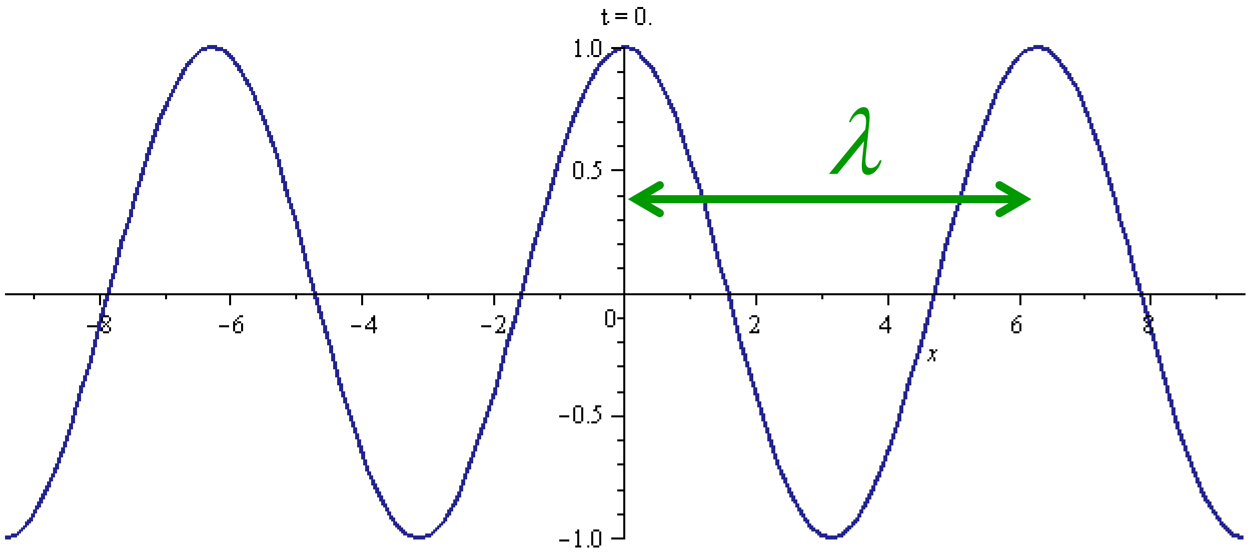


Light waves through a big hole

$$\theta \approx \frac{l}{\lambda}$$



Sound waves through a small hole



Simple Waves

- cos and sin have periodicity 2π
- If you increase kx by 2π, wave will look the same
- If you increase ωt by 2π, wave will look the same

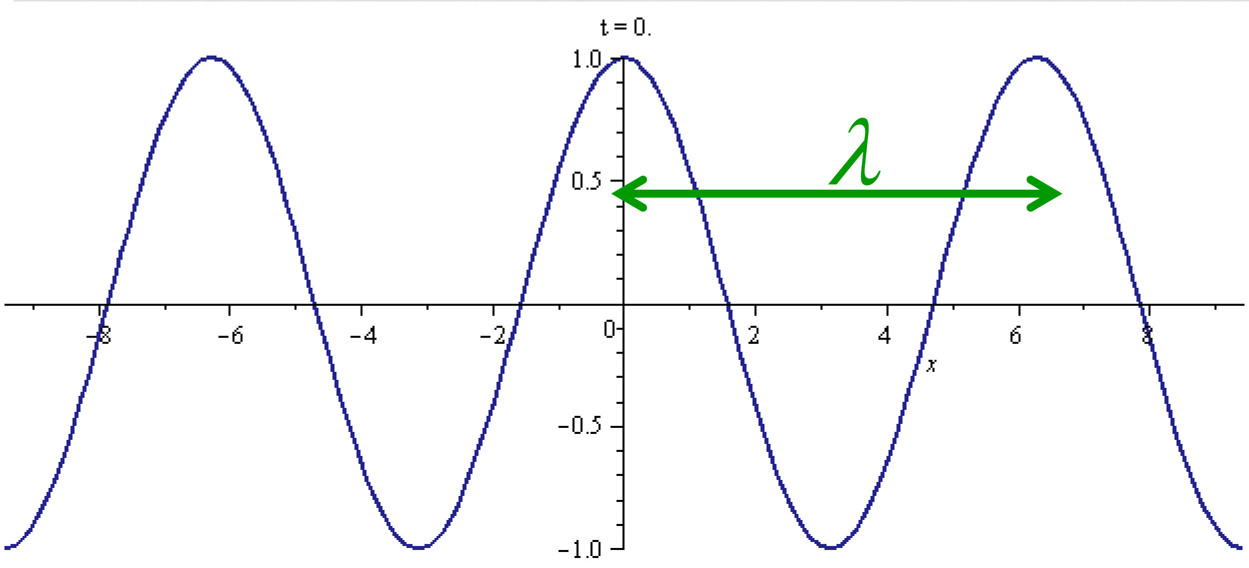
- Simple waves look like cosines or sines:
- k is called the wave number
  - Units of inverse meters
- ω is called the angular frequency
  - Units of inverse seconds
- Wavelength λ is how far you have to go in space before it repeats
  - Related to wave number k
- Period T is how long you have to wait in time before it repeats
  - Related to angular frequency ω
- Frequency f is how many times per second it repeats
  - The reciprocal of period

$$\psi(x,t) = A\cos(kx - \omega t)$$
$$\psi(x,t) = A\sin(kx - \omega t)$$

$$\lambda = 2\pi/k$$

$$\omega = 2\pi/T = 2\pi f$$

Phase velocity



$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$k\lambda = 2\pi$$

$$v_p = \lambda f = \frac{\omega}{k}$$

- The wave moves a distance of one wavelength λ in one period T
- From this, we can calculate the *phase velocity* denoted v_p
  - It is how fast the peaks and valleys move

$$v_p = \frac{\lambda}{T} = \lambda f = \frac{2\pi}{k}\frac{\omega}{2\pi} = \frac{\omega}{k}$$

$$v_p = \frac{\omega}{k} = \frac{ck}{k} = c$$

Adding two waves

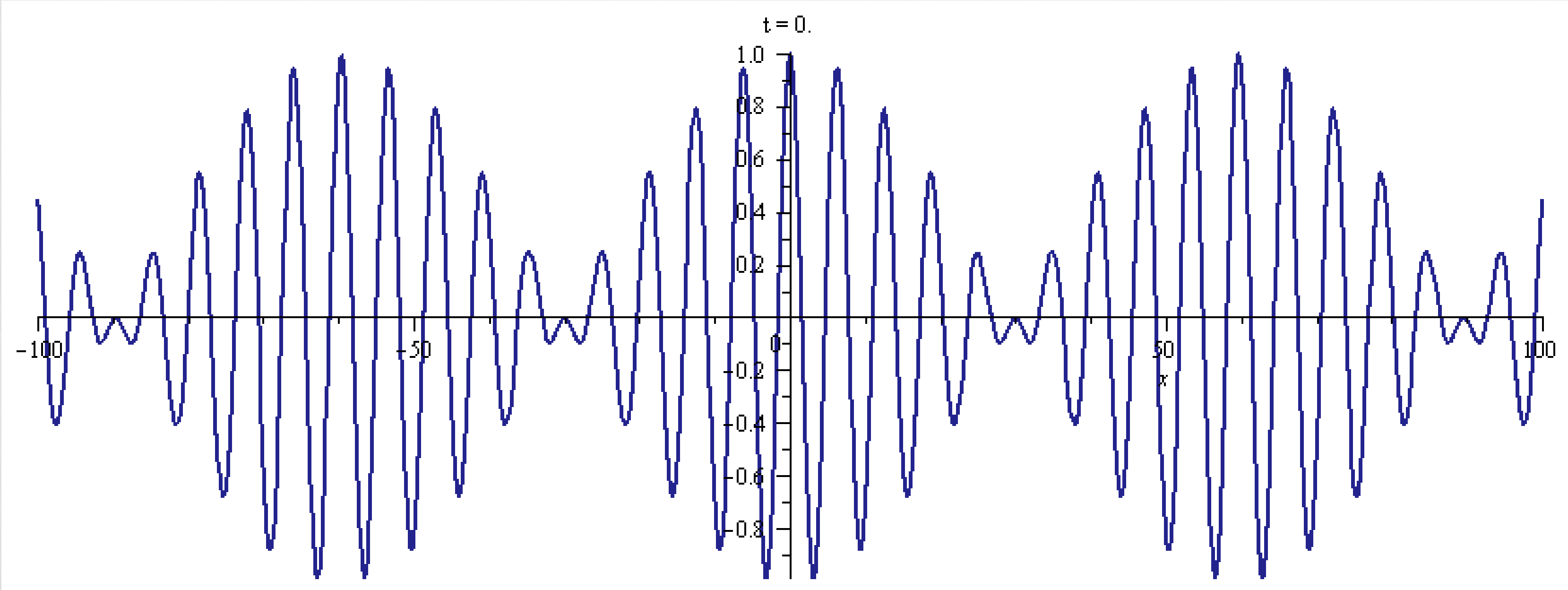
•Real waves are almost always combinations of multiple wavelengths

•Average these two expressions to get a new wave:

$$\psi_1 = \cos\left(k_1 x - \omega_1 t\right)$$

$$\psi_2 = \cos\left(k_2 x - \omega_2 t\right)$$

$$\psi\left(x,t\right) = \tfrac{1}{2}\cos\left(k_1 x - \omega_1 t\right) + \tfrac{1}{2}\cos\left(k_2 x - \omega_2 t\right)$$



•This wave has two kinds of oscillations:

- •The oscillations at small scales
- •The “lumps” at large scales

Analyzing the sum of two waves:

$$\psi(x,t) = \tfrac{1}{2}\cos(k_1 x - \omega_1 t) + \tfrac{1}{2}\cos(k_2 x - \omega_2 t)$$

Need to derive some obscure trig identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

•Average these:

$$\tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta) = \cos\alpha\cos\beta$$

•Substitute:

$$\alpha = \tfrac{1}{2}(A+B)$$

$$\beta = \tfrac{1}{2}(A-B)$$

$$\tfrac{1}{2}\cos A + \tfrac{1}{2}\cos B = \cos\left[\tfrac{1}{2}(A+B)\right]\cos\left[\tfrac{1}{2}(A-B)\right]$$

Rewrite wave function:

$$\psi(x,t) = \underbrace{\cos(\bar{k}x - \bar{\omega}t)}_{\text{Small scale oscillations}}\underbrace{\cos(\Delta k\cdot x - \Delta\omega\cdot t)}_{\text{Large scale oscillations}}$$

$$\bar{k} = \tfrac{1}{2}(k_1 + k_2)$$

$$\bar{\omega} = \tfrac{1}{2}(\omega_1 + \omega_2)$$

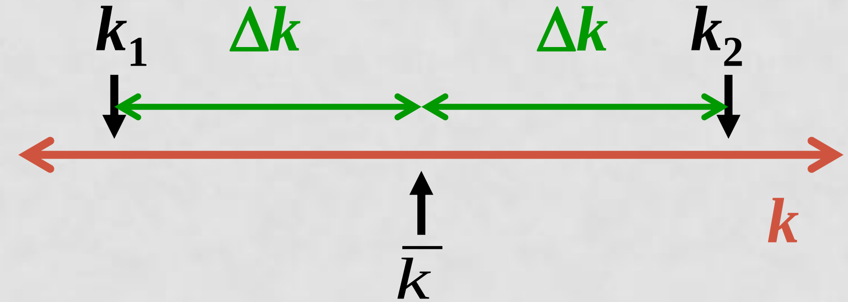
Small scale oscillations

Large scale oscillations

$$\Delta k = \tfrac{1}{2}(k_1 - k_2)$$

$$\Delta\omega = \tfrac{1}{2}(\omega_1 - \omega_2)$$

The "uncertainty" of two waves

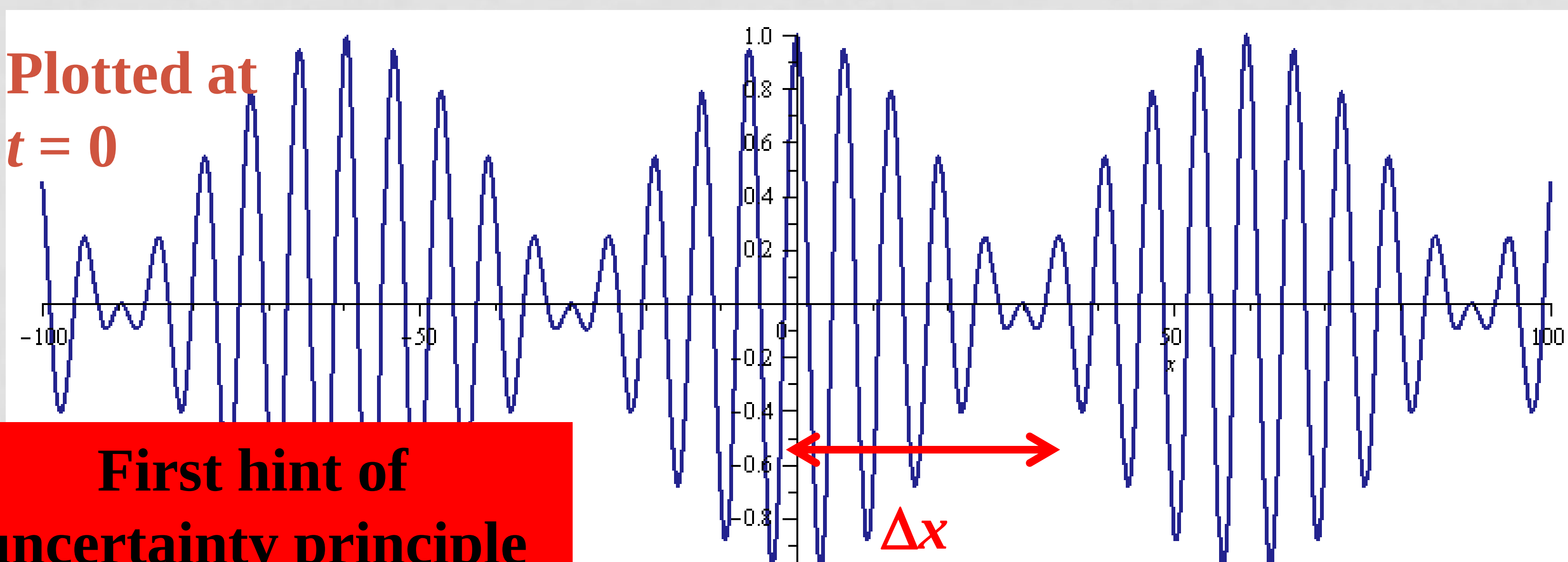


Our wave is made of two values of *k*:

- $\bar{k}$ is the average value of these two
- Δk is the amount by which the two values of *k* actually vary from $\bar{k}$
  - The value of *k* is uncertain by an amount Δk

- Each "lump" is spread out in space also
- Define Δx as the distance from the center of a lump to the edge
- The distance is where the cosine vanishes

Plotted at $t = 0$



$$\cos(\Delta k \Delta x) = 0$$

$$\Delta k \Delta x = \tfrac{1}{2}\pi$$

$$\Delta k \Delta x \;\square\; 1$$

First hint of uncertainty principle

Group Velocity

$$\psi(x,t) = \cos(\bar{k}x - \bar{\omega}t)\cos(\Delta k \cdot x - \Delta\omega \cdot t)$$

Small scale oscillations

Large scale oscillations

The velocity of little oscillations governed by the first factor

- Leads to the same formula as before for phase velocity:

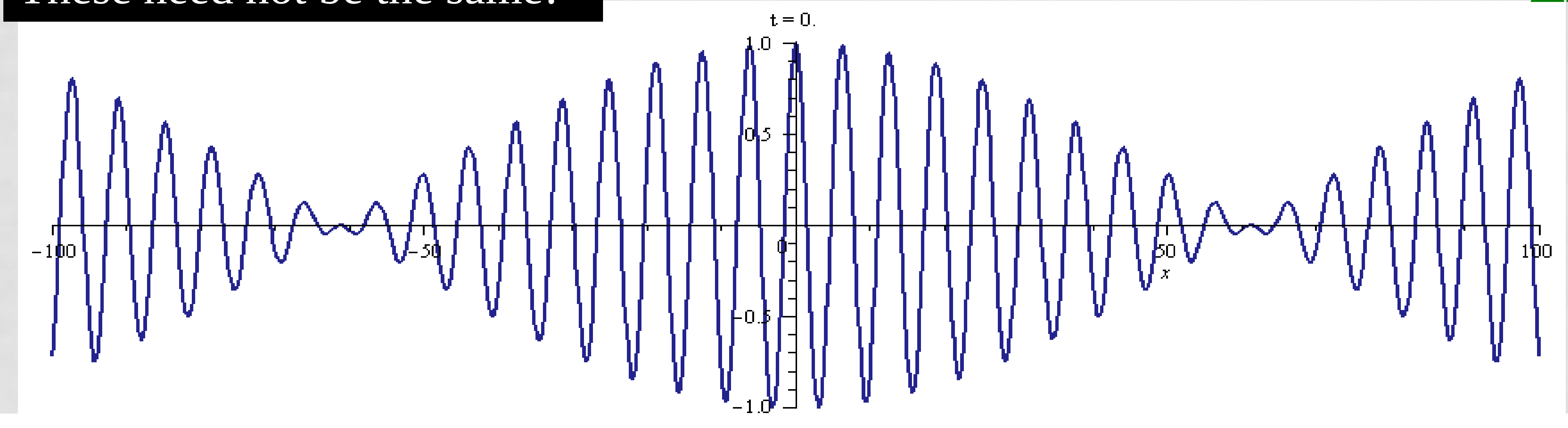
$$v_p = \frac{\bar{\omega}}{\bar{k}}$$

The velocity of big oscillations governed by the second factor

- Leads to a formula for group velocity:

$$v_g = \frac{\Delta\omega}{\Delta k}$$

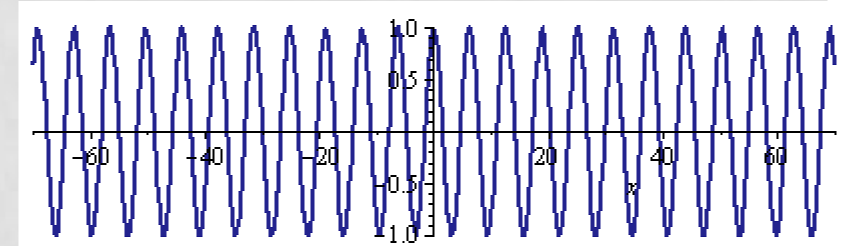
These need not be the same!



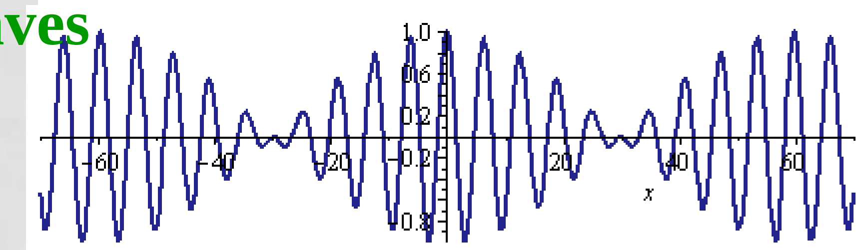
More Waves

- Two waves allow you to create localized "lumps"
- Three waves allow you to start separating these lumps
- More waves lets you get them farther and farther apart
- Infinity waves allows you to make the other lumps disappear to infinity – you have one lump, or a wave packet
- A single lump – a wave packet – looks and acts a lot like a particle

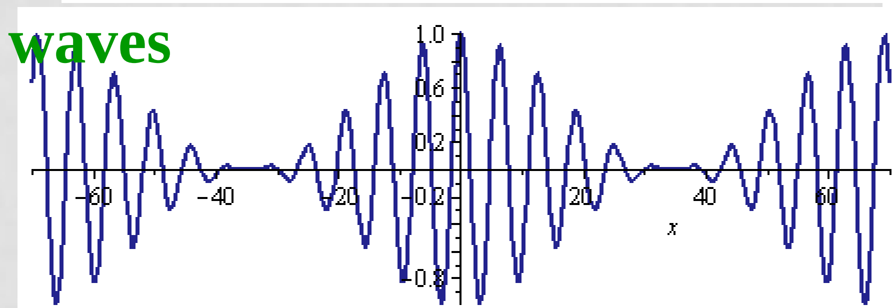
One wave



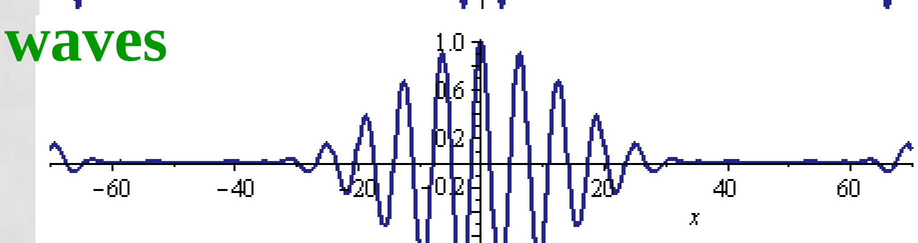
Two waves



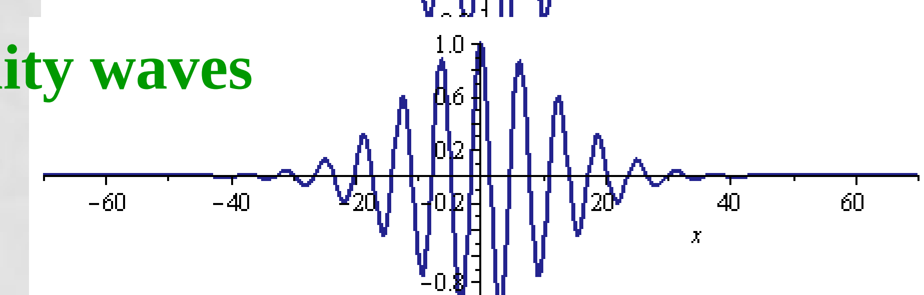
Three waves



Five waves

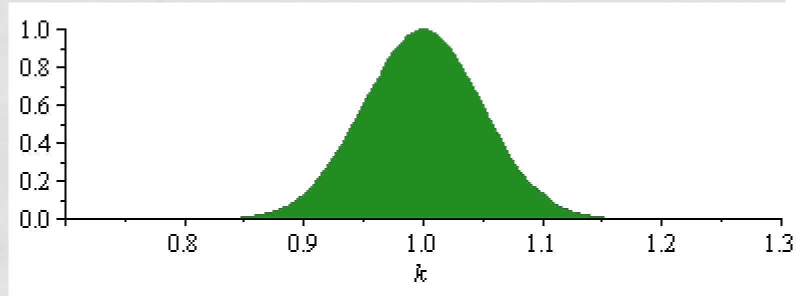


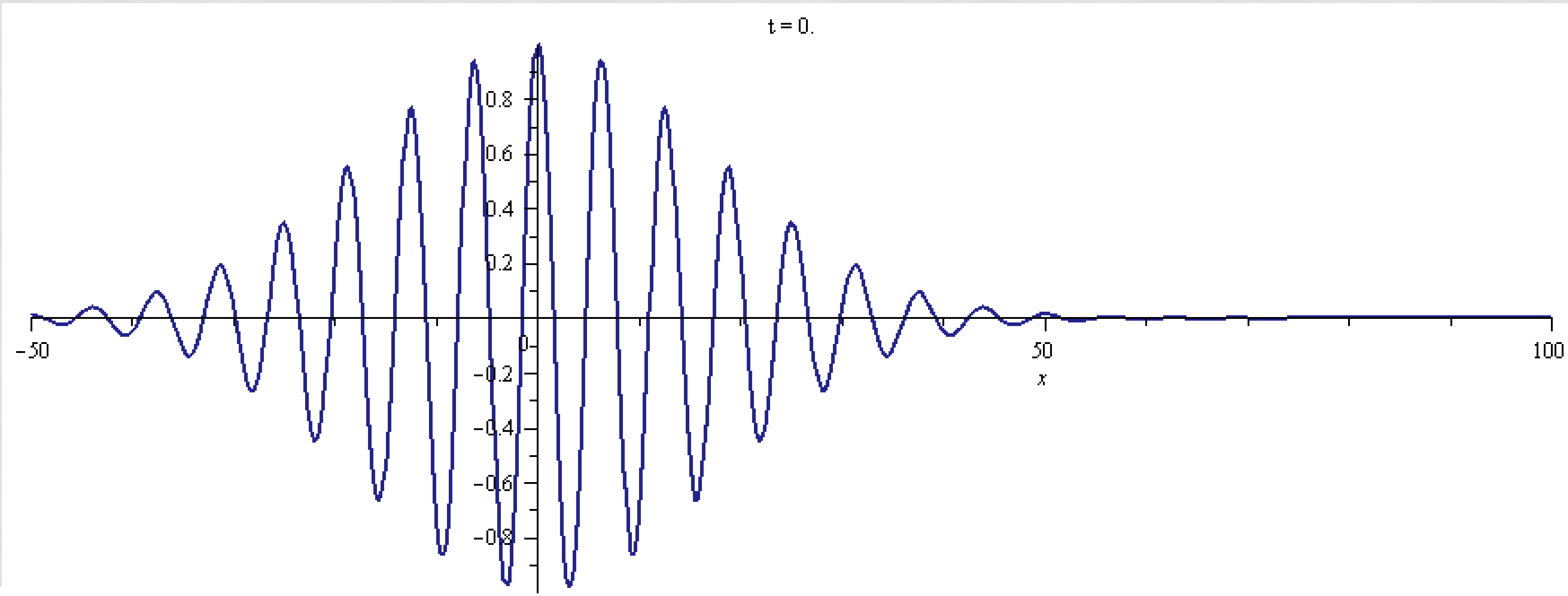
Infinity waves



Wave Packets

- We can combine many waves to separate a “lump” from its neighbors
- With an infinite number of waves, we can make a wave packet
  - Contains continuum of wave numbers *k*
- Resulting wave travels and mostly stays together, like a particle
- Note both *k*-values and *x*-values have a spread Δk and Δx.





Phase and Group velocity

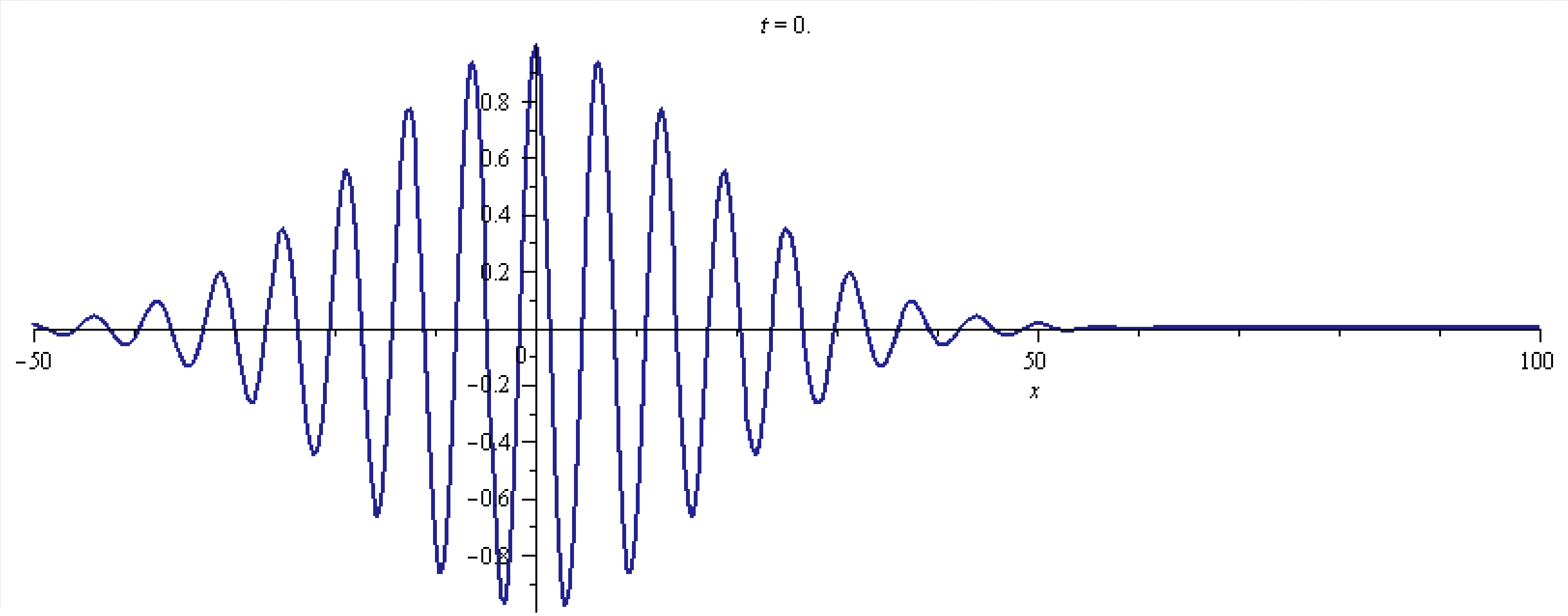
Compare to two wave formulas:

•Phase velocity formula is exactly the same, except we simply rename the average values of *k* and ω as simply *k* and ω

•Group velocity now involves very closely spaced values of *k* (and ω), and therefore we rewrite the differences as . . .

$$v_p = \frac{\bar{\omega}}{\bar{k}} \Rightarrow \boxed{v_p = \frac{\omega}{k}}$$

$$v_g = \frac{\Delta\omega}{\Delta k} \Rightarrow \boxed{v_g = \frac{d\omega}{dk}}$$



Phase and Group velocity

$$v_p = \frac{\omega}{k}$$

How to calculate them:

- You need the *dispersion relation:* the relationship between ω and k, with only constants in the formula
- Example: light in vacuum has $\omega = ck$

$$v_p = \frac{\omega}{k} = \frac{ck}{k} = c$$

$$v_g = \frac{d\omega}{dk}$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(ck) = c$$

Theorem: Group velocity doesn't always equal phase velocity

$$\omega = kv_p$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(v_p k) = v_p + \frac{dv_p}{dk}k \neq v_p$$

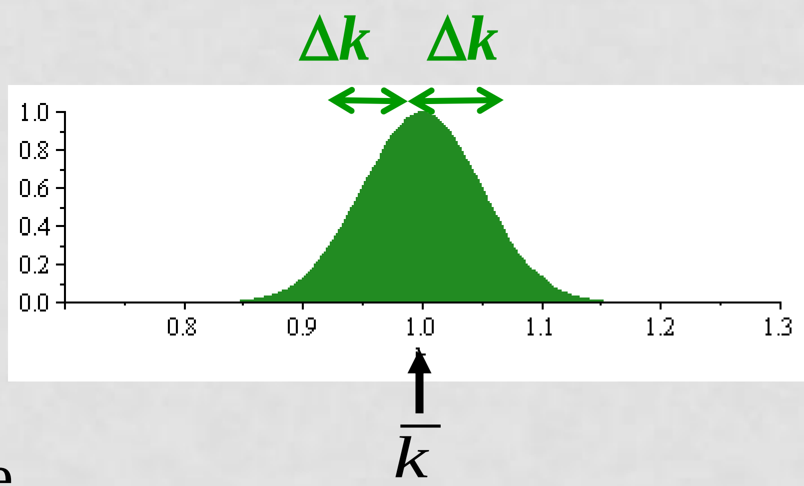
If the dispersion relation is $\omega = Ak^2$, with A a constant, what are the phase and group velocity?

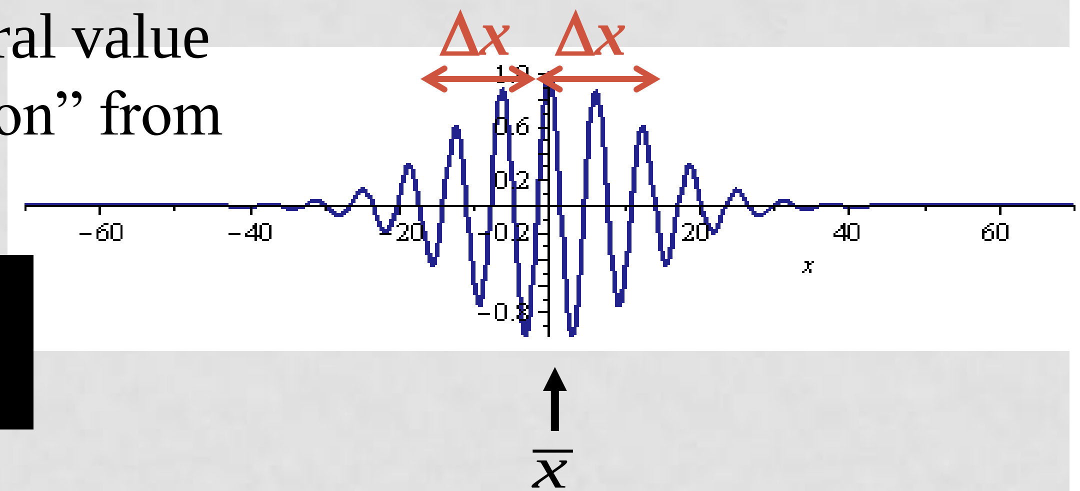
$$v_p = \frac{\omega}{k} = \frac{Ak^2}{k} = Ak$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(Ak^2) = 2Ak$$

The Classical Uncertainty Principle

- The wave number of a wave packet is not exactly one value
  - It can be approximated by giving the central value
  - And the uncertainty, the "standard deviation" from that value
- The position of a wave packet is not exactly one value
  - It can be approximated by giving the central value
  - And the uncertainty, the "standard deviation" from that value





These quantities are related:
- Typically, $\Delta x \, \Delta k \sim 1$

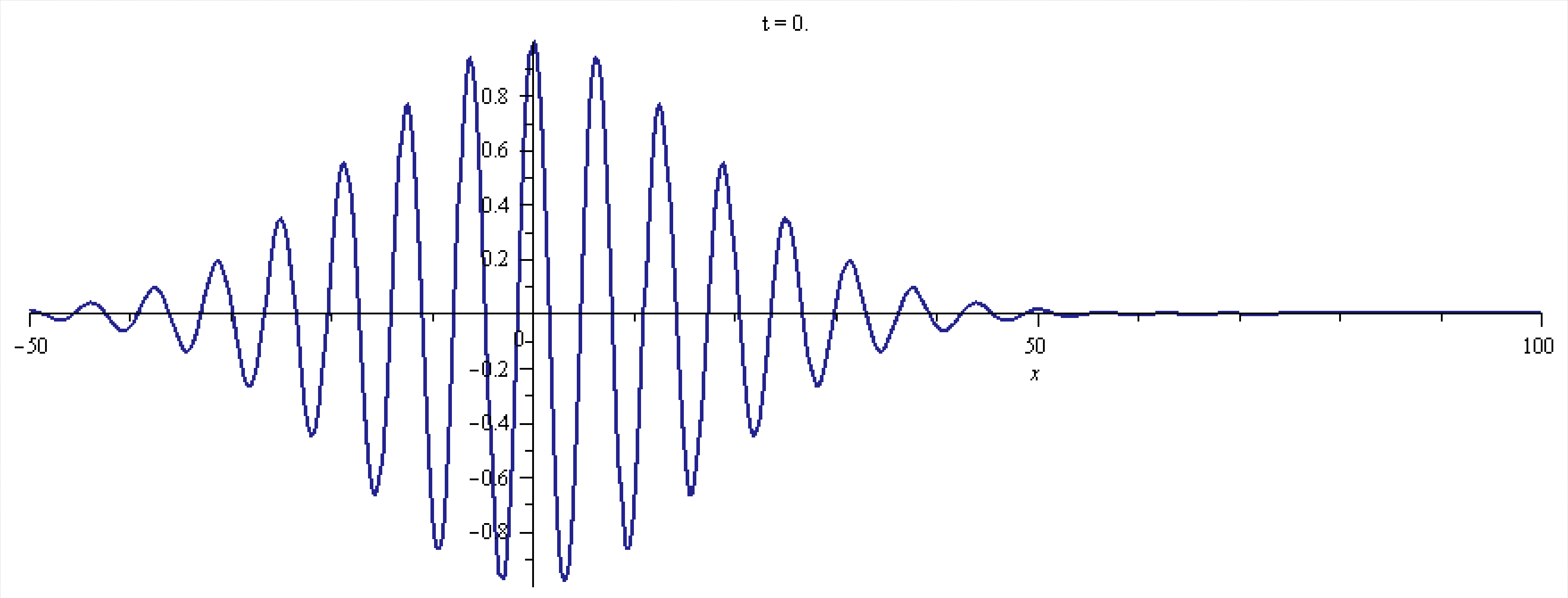
Precise Relation:
(proof hard)

$$\Delta x \Delta k \geq \tfrac{1}{2}$$

Uncertainty in the Time Domain

Stand and watch a wave go by at one place

- You will see the wave over a period of time Δt
- You will see the wave with a combination of angular frequencies $\Delta\omega$
- The same uncertainty relationship applies in this domain



$$\Delta t \Delta \omega \geq \frac{1}{2}$$